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Recovery of Singularities for Formally Determined Inverse Problems

Ziqi Sun[1]* and Gunther Uhlmann[2]**

[1] Department of Mathematics and Statistics, Wichita State University, Wichita, Kansas 67208, USA

[2] Department of Mathematics, University of Washington, Seattle, Washington 98195, USA

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Abstract. In this paper we considered several formally determined problems in two dimensions. There are no global identifiability results for these problems. However, we can recover an important feature of these functions, namely their singularities. More precisely, we prove that one can determine the location and strength of singularities of an L^∞ compactly supported potential by knowing the associated scattering amplitude at a fixed energy. Also we prove that one can determine the location and strength of the singularities of the sound speed of a medium by making measurements just on the boundary of the medium.

1. Introduction and Statement of the Results

In this paper we consider formally determined inverse problems in two dimensions. These problems involve determination of the sound speed of a medium by making measurements at the boundary of the medium or a quantum mechanical potential by making scattering measurements away from the support of the potential.

For the problems under consideration there are no global identifiability results available in the case of a general L^∞ potential or sound speed. The results known are either local ([S-U, Su I, II]) or generic ([Su-U I]). Kohn and Vogelius ([K-V]) proved a global identifiability result in the case that the potential is piecewise analytic. In this paper we consider the problem of determining the strength and location of the singularities of the sound speed or the potential from either boundary measurements or scattering information.

All the results we prove are reduced to prove a similar result for the inverse problem of determining a bounded, measurable potential from the Dirichlet to Neumann map associated to the Schrödinger equation at zero energy. We proceed next to define this map.

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Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain with smooth boundary. Let $q \in L^\infty(\Omega)$. Assume that 0 is not a Dirichlet eigenvalue of $-\Delta + q$. Then $\forall f \in H^{\frac{1}{2}}(\partial\Omega)$ there is a unique solution $u \in H^1(\Omega)$ of the Dirichlet problem

$$(-\Delta + q)u = 0 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = f. \tag{1.1}$$

The Dirichlet to Neumann map is defined by

$$\Lambda_q(f) = \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} \tag{1.2}$$

with u solution of (1.1) and ν denotes the unit outer normal to $\partial\Omega$. The inverse problem we consider is to determine the injectivity of the map

$$q \overset{\Lambda}{\mapsto} \Lambda_q . \tag{1.3}$$

It is known that Λ is locally injective near $q = 0$ ([S-U]) or $q =$ constant ([Su I]), locally injective in an open and dense set of potentials in $W^{1,\infty}(\Omega)$ ([Su-U I]) and globally injective on pairs of potentials in an open and dense set in $W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega)$ ([Su-U I]). In [Su-U II] the authors proved that one can determine the strength and location of singularities of potentials q having jump type singularities across a subdomain of Ω from Λ_q. In this paper we extend this result to determining *general* L^∞ singularities of potentials from Λ_q. More precisely:

Theorem A. *Let $q_i \in L^\infty(\Omega)$ with 0 not a Dirichlet eigenvalue of $-\Delta + q_i$, $i = 1, 2$. Assume*

$$\Lambda_{q_1} = \Lambda_{q_2} .$$

Then

$$q_1 - q_2 \in C^\alpha(\bar{\Omega}) \ \forall \alpha, \quad 0 \leqq \alpha < 1 .$$

Here C^α denotes the Hölder space of order α. We first apply Theorem A to the inverse scattering problem by a potential at a fixed energy. We describe the problem below. The scattering amplitude of a potential $q \in L^\infty(\mathbb{R}^2)$ with compact support is defined via the outgoing eigenfunctions. Namely, for $\lambda \in \mathbb{R} - 0, \theta, \omega \in S^1$ there exists $\psi_+(\lambda, x, \omega)$ solution of

$$(-\Delta + q - \lambda^2)\psi_+ = 0 \tag{1.4}$$

satisfying

$$\psi_+ = e^{i\lambda x\cdot\omega} + \frac{a_q(\lambda, \theta, \omega)e^{i\lambda|x|}}{|x|^{\frac{1}{2}}} + 0(|x|^{-\frac{3}{2}}) \tag{1.5}$$

with $\theta = \frac{x}{|x|}$. The scattering amplitude, $a_q(\lambda, \theta, \omega)$ measures the effect of the potential q on plane waves of the form $e^{i\lambda x\cdot\omega}$. The inverse scattering problem at a fixed energy λ, in two dimensions, is to determine the potential q from the scattering amplitude $a_q(\lambda, \theta, \omega)$ with λ fixed and with $\theta, \omega \in S^1$. In this paper we prove:

Theorem B. *Let $q_1, q_2 \in L^\infty(\mathbb{R}^2)$ with compact support. If*

$$a_{q_1}(\lambda, \theta, \omega) = a_{q_2}(\lambda, \theta, \omega)$$

for all $\theta, \omega \in S^1$ for a fixed λ, then

$$q_1 - q_2 \in C^\alpha(\mathbb{R}^2) \ \forall\alpha, \quad 0 \leqq \alpha < 1 .$$

We now apply Theorem A to an inverse problem arising in geophysics: to determine the sound speed of a medium by making measurements at the boundary. We formulate more precisely the problem. Let Ω be a bounded region with smooth boundary. We denote by $c(x)(>0)$ the sound speed of the medium Ω and assume that $c(x) = c_0 > 0$ for $x \in \mathbb{R}^2 - \Omega$. The scattered pressure field generated by a point source at a point $x_0 \in \partial\Omega$ is given by the outgoing Green's kernel which satisfies the outgoing radiation condition and solves

$$\Delta \mathscr{G}_c(x, x_0, \lambda) + \frac{\lambda^2}{c^2(x)} \mathscr{G}_c(x, x_0, \lambda) = -\delta(x - x_0) . \tag{1.6}$$

The inverse problem is to determine $c(x)$ by measuring $\mathscr{G}_c(x, x_0, \lambda)$ with $x, x_0 \in \partial\Omega$ and λ fixed. In this paper we prove:

Theorem C. *Let $c_1, c_2 \in L^\infty(\mathbb{R}^2)$, $c_j > 0$, $j = 1, 2$, $c_1(x) = c_2(x) = c_0 > 0$* for *$x \in \mathbb{R}^2 - \Omega$, where c_0 is a constant. If*

$$\mathscr{G}_{c_1}(x, x_0, \lambda) = \mathscr{G}_{c_2}(x, x_0, \lambda)$$

for $x, x_0 \in \partial\Omega$ and λ fixed. Then

$$c_1 - c_2 \in C^\alpha(\mathbb{R}^2), \ \forall\alpha, \quad 0 \leqq \alpha < 1 .$$

The proof of Theorem A uses the special solutions constructed in [S-U]. Namely, there are solutions of

$$(-\Delta + q)u = 0 \quad \text{in } \mathbb{R}^2$$

(by extending $q = 0$ in $\mathbb{R}^2 - \Omega$) of the form

$$u = e^{x\zeta}(1 + \omega(x, \zeta)) \tag{1.7}$$

with $\zeta \in \mathbb{C}^2, \zeta \cdot \zeta = 0$ and ψ decaying at ∞ for $|\zeta| > K$ (see Proposition 2.1 for a more precise statement).

Then we can define the function T_q considered by Beals and Coifman ([B-C]) and Ablowitz and Nachman ([N-A]) in the $\bar\partial$ approach to inverse scattering.

$$T_q(k) = \Phi_K(k) \int_\Omega e^{ixk} q(x)(1 + \omega(x, \zeta))dx ,$$

where

$$\zeta = \frac{1}{2}(ik + Jk), \quad k = (k_1, k_2) \in \mathbb{R}^2, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad i = \sqrt{-1} ,$$

and

$$\Phi_K(k) = \begin{cases} 1 & \text{for } |k| \geqq K \\ 0 & \text{for } |k| < K . \end{cases}$$

The main point is the following result (see Theorem 3.1).

Theorem D. *Let* $q \in L^\infty(\mathbb{R}^2)$, $\operatorname{supp} q \subseteq \Omega$. *Then*

$$q - \mathscr{F}^{-1} T_q \in C^\alpha(\bar{\Omega}) \ \forall \alpha, \quad 0 \leqq \alpha < 1\,,$$

where $\mathscr{F}$ *denotes the Fourier transform.*

Theorem A then follows from Theorem D and the fact that

Theorem E. ([Su-U II]) *Let* $q_i \in L^\infty(\Omega)$ *as in Theorem A. Extend* $q_i = 0$ *on* $\mathbb{R}^2 - \Omega$. *Then*

$$\Lambda_{q_1} = \Lambda_{q_2}$$

implies that

$$T_{q_1} = T_{q_2}\,.$$

Theorem D suggests a reconstruction method to determine the location and the singularities of q from Λ_q: Using the methods of Nachman ([N]), one can reconstruct T_q from Λ_q and by Theorem D, the singularities of q.

In Sect. 2 we develop the preliminaries. In Sect. 3 we prove Theorem A. In Sect. 4 we reduce the proofs of Theorem B and C to Theorem A. We also state in Sect. 4 extensions of the results to other classes of L^p potentials.

2. Preliminaries

In what follows we use the following notation:

$$L^\infty_\Omega = \{ f \in L^\infty, \operatorname{supp} f \subseteq \bar{\Omega} \}\,,$$

$$L^p_\delta = \{ f, (1 + |x|^2)^{\frac{\delta}{2}} f \in L^p(\mathbb{R}^2) \}\,.$$

Proposition 2.1. ([S-U]) *Let* $\zeta \in \mathbb{C}^2$ *with* $\zeta \cdot \zeta = 0$. *Let* $p > 1$, $\delta \in \mathbb{R}$ *with* $-1 < \delta - 1 + \frac{2}{p} < 0$. *Let* $q \in L^\infty_\Omega$. *Then there exists a constant* $K = K(\Omega, p, \delta, \|q\|_{L^\infty(R^2)})$ *such that for* $|\zeta| > K$, *there exists a unique solution of* $(-\Delta + q)u = 0$ *in* $\mathbb{R}^2$ *of the form*

$$u(x, \zeta) = e^{x \cdot \zeta}(1 + \omega(x, \zeta)) \tag{2.1}$$

with

$$\omega \in L^p_\delta(\mathbb{R}^2)\,.$$

Furthermore, there exists a constant $C = C(\Omega, p, \delta, \|q\|_{L^\infty(\mathbb{R}^2)})$ *such that*

$$\|\omega\|_{L^p_\delta(\mathbb{R}^2)} \leqq \frac{C}{|\zeta|}\,. \tag{2.2}$$

If we choose

$$\zeta = \frac{1}{2}(ik + Jk), \quad k = (k_1, k_2) \in \mathbb{R}^2, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad i = \sqrt{-1}\,,$$

then a straightforward computation shows that

$$\bar{\partial}(\partial + (k_2 + ik_1))\omega - q\omega = q\,, \tag{2.3}$$

where

$$\bar{\partial} = \left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right), \quad \partial = \left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right).$$

The above proposition follows directly from the lemma below.

We shall write $\psi(x,k)$ instead of $\psi(x,\zeta)$ from now on.

Lemma 2.2. ([S-U]) *Let $q \in L^\infty_\Omega$, $f \in L^p_{\delta+1}(\mathbb{R}^2)$, $p > 1$, $\delta \in \mathbb{R}$ with $-1 < \delta - 1 + \frac{2}{p} < 0$. Then there exists a constant $K = k(\Omega, p, \delta, \|q\|_{L^\infty(\mathbb{R}^2)})$ such that for $|k| > K$ there exists a unique function $\omega(x,k) \in L^p_\delta(\mathbb{R}^2)$ satisfying*

$$\bar{\partial}(\partial + k_2 + ik_1)\omega - q\omega = f \quad \text{in } \mathbb{R}^2 . \tag{2.4}$$

Moreover,

$$\|\omega\|_{L^p_\delta(\mathbb{R}^2)} \leqq \frac{C}{|k|} \|f\|_{L^2_{\delta+1}(\mathbb{R}^2)} ,$$

where $C = C(\delta, p, \|q\|_{L^\infty(\mathbb{R}^2)})$ is a constant.

The proof of Lemma 2.2 can be done by using an iteration procedure and reducing it to a special case $q = 0$ in (2.4). We write

$$\omega = \sum_{j=0}^{\infty} \omega_j \tag{2.5}$$

$$\begin{cases} \bar{\partial}(\partial + (k_2 + ik_1))\omega_0 = f \\ \bar{\partial}(\partial + (k_2 + ik_1))\omega_{j+1} = q\omega_j, \quad j \geqq 1 . \end{cases} \tag{2.6}$$

Assuming Lemma 2.2 holds when $q = 0$, one sees immediately that

$$\|\omega_j\|_{L^p_\delta(\mathbb{R}^2)} \leqq \left(\frac{C\|q\|_{L^\infty(R^2)}}{|k|}\right)^{j+1} \|f\|_{L^p_{\delta+1}(\mathbb{R}^2)}, \quad j \geqq 0 , \tag{2.7}$$

which shows that $\sum \omega_j$ converges in $L^p_\delta(\mathbb{R}^2)$ for large $|k|$.

The equation (2.4) with $q = 0$ can be reduced to $\bar{\partial}$ equations. A direct computation shows that

$$\omega = \frac{a + e^{-ixk}b}{k_2 + ik_1} \tag{2.8}$$

solves (2.4) with $q = 0$ if a and b solve the following $\bar{\partial}$ equations:

$$\begin{cases} \bar{\partial}a = f \\ \partial b = -e^{ixk}\partial a \end{cases} . \tag{2.9}$$

The following result shows that the equation $\bar{\partial}u = f$ is unique solvable in the space $L^p_\delta(\mathbb{R}^2)$. In what follows we denote by $\bar{\partial}^{-1}$ the inverse of $\bar{\partial}$ in $L^p_\delta(\mathbb{R}^2)$. The reference for the next results is the paper by Nirenberg and Walker ([N-W]).

Proposition 2.3. *Let $p > 1$, $\delta \in \mathbb{R}$ with $-1 < \delta - 1 + \frac{2}{p} < 0$. Then*

$$(\bar{\partial}^{-1}f)(z) = -\frac{1}{\pi} \int_{\mathbb{R}^2} \frac{f(\xi)}{\xi - z} d\xi \tag{2.10}$$

defines a bounded operator from $L^p_{\delta+1}$ *to* L^p_δ. *Moreover, the following pointwise estimate holds*:

$$|\bar{\partial}^{-1}f(z)| \leqq C(|z|+1)^{-\frac{2}{p}-\delta}\|f\|_{L^p_{\delta+1}}\,. \tag{2.11}$$

If, in addition, $f \in L^\infty_\Omega$, *then*

$$|\bar{\partial}^{-1}f(z)| \leqq C(|z|+1)^{-1}\,\|f\|_{L^\infty_\Omega}\,, \tag{2.12}$$

where $z = x_1 + ix_2$.

The next result gives properties of the singular integral operator $\partial\bar{\partial}^{-1}$.

Proposition 2.4. *Let* $p > 1$, $\delta \in \mathbb{R}$, *with* $-1 < \delta - 1 + \frac{2}{p} < 0$. *Then*

$$\partial\bar{\partial}^{-1}f(z) = -\frac{1}{\pi}\,\text{p.v.}\int_{\mathbb{R}^2}\frac{f(\xi)}{(\xi - z)^2}\,d\xi \tag{2.13}$$

defines a bounded operator from $L^p_{\delta+1}$ *to* $L^p_{\delta+1}$. *If, in addition,* $f \in L^\infty_\Omega$, *then*

$$\|\partial\bar{\partial}^{-1}f\|_{L^p(\mathbb{R}^2)} \leqq C\|f\|_{L^\infty_\Omega}, \quad C = C(p)\,. \tag{2.14}$$

Also, the following pointwise estimate holds for large $|z|$:

$$|\partial\bar{\partial}^{-1}f(z)| \leqq \frac{C}{(1+|z|)^2}\,\|f\|_{L^\infty_\Omega}\,. \tag{2.15}$$

One sees that Propositions 2.3 and 2.4 give unique solutions a and b in L^p_δ in (2.9).

We now turn to the T_q function. Given $q \in L^\infty_\Omega$, we define

$$T_q(k) = \Phi_K(k)\int_\Omega e^{ixk}q(x)(1+\omega(x,k))dx\,, \tag{2.16}$$

where ω is the unique L^p_δ solution to (2.3) and

$$\Phi_K(k) = \begin{cases} 1 & |k| > K \\ 0 & |k| \leqq K \end{cases}$$

with K as in Lemma 2.2.

The function T_q is the two dimensional analogue of the scattering transform considered in the $\bar{\partial}$ approach to the inverse scattering problem by Beals and Coifman [B-C] and Ablowitz and Nachman [N-A]. An important fact about T_q is that knowledge of Λ_q or the scattering amplitude at the fixed energy determines T_q uniquely as a function of k [Su-U II]. Thus, the question of determining the discontinuities of q from Λ_q or from the scattering amplitude at a fixed energy is reduced to determining the discontinuities of q from T_q.

3. Proof of Theorem A

The proof of Theorem A is reduced to the proof of the following

Theorem 3.1. *Let* $q \in L^\infty_\Omega$ *and let* T_q *be the function defined in* (2.16), *then*

$$q - \mathscr{F}^{-1}T_q|_\Omega \in C^\alpha(\bar{\Omega}), \quad 0 \leqq \alpha < 1\,, \tag{3.1}$$

where $\mathscr{F}^{-1}$ *denotes the inverse Fourier transform.*

Using the expansion (2.5) we may write T_q as

$$T_q = \Phi_K \mathscr{F} q + \sum_{j=0}^{\infty} T_q^{(j)} , \tag{3.2}$$

where $T_q^{(j)}, 0 \leqq j < \infty$ are function of k given by

$$T_q^{(j)}(k) = \Phi_K(k) \int_{\Omega} e^{ixk} q(x) \omega_j(x, k) dx , \tag{3.3}$$

where ω_j is given by (2.6) with $f = q$.

Applying the estimate (2.7) to (3.3) yields an estimate for $T_q^{(j)}$,

$$|T_q^{(j)}(k)| \leqq \frac{C^{j+1}}{|k|^{j+1}}, \quad 0 \leqq j < \infty , \tag{3.4}$$

where $C = C(\Omega, \|q\|_{L^\infty(\mathbb{R}^2)})$, which implies that

$$\mathscr{F}^{-1} T_q^{(j)} \in H^{j-\varepsilon}(\mathbb{R}^2), \quad \forall \varepsilon > 0 . \tag{3.5}$$

This result is sufficient to conclude that

$$\mathscr{F}^{-1}\left(\sum_{j=2}^{\infty} T_q^{(j)}\right)\Big|_{\Omega} \in C^{1-\varepsilon}(\bar{\Omega}) \quad \forall \varepsilon > 0 .$$

However, for $T_q^{(0)}$ and $T_q^{(1)}$, (3.5) is too weak. We shall show that $\mathscr{F}^{-1} T_q^{(0)}$ and $\mathscr{F}^{-1} T_q^{(1)}$ are actually in the C^α class with $0 \leqq \alpha < 1$.

Proposition 3.2. $\mathscr{F}^{-1} T_q^{(0)}|_\Omega$ and $\mathscr{F}^{-1} T_q^{(1)}|_\Omega$ *belongs to* $C^\alpha(\mathbb{R}^2)$ *for any* α *with* $0 \leqq \alpha < 1$.

Using (3.5) together with Proposition 3.2 we can prove Theorem 3.1.

Proof of Theorem 3.1. We have

$$\begin{aligned} q - \mathscr{F}^{-1} T_q &= q - \mathscr{F}^{-1} \Phi_K \mathscr{F} q + \sum_{j=0}^{\infty} \mathscr{F}^{-1} T_q^{(j)} \\ &= -\mathscr{F}^{-1}(\Phi_K - 1)\mathscr{F} q + \mathscr{F}^{-1} T_q^{(0)} + \mathscr{F}^{-1} T_q^{(1)} \\ &\quad + \sum_{j=2}^{\infty} \mathscr{F}^{-1} T_q^{(j)} . \end{aligned}$$

Since $\Phi_K - 1$ is supported in $\{|k| < K\}$, $\mathscr{F}^{-1}(\Phi_K - 1)\mathscr{F} q \in C^\infty(\mathbb{R}^2)$. Using Sobolev's embedding theorem we obtain $\mathscr{F}^{-1} T_q^{(j)} j \geqq 2$ and thus $\sum_{j=2}^{\infty} \mathscr{F}^{-1} T_q^{(j)}$ belongs to $C^{1-\varepsilon}(\mathbb{R}^2)$ for any ε with $0 < \varepsilon \leqq 1$. Therefore, Theorem 3.1 follows from Proposition 3.2. □

The rest of this section is devoted to the proof of Proposition 3.2. A direct computation based on (2.6)–(2.9) yields the explicit formulas for ω_0 and ω_1 given below:

$$\omega_0 = (k_2 + ik_1)^{-1}[\bar{\partial}^{-1} q - e^{-ixk} \partial^{-1}(e^{ixk} \partial \bar{\partial}^{-1} q)] , \tag{3.6}$$

$$\begin{aligned} \omega_1 &= (k_2 + ik_1)^{-2}[\bar{\partial}^{-1}(q \bar{\partial}^{-1} q) - \bar{\partial}^{-1}(q e^{-ixk} \partial^{-1}(e^{ixk} \partial \bar{\partial}^{-1} q))] \\ &\quad - (k_2 + ik_1)^{-2} e^{-ixk} \partial^{-1}[e^{ixk} \partial \bar{\partial}^{-1}(q \bar{\partial}^{-1} q)] \\ &\quad - (k_2 + ik_1)^{-2} e^{-ixk} \partial^{-1}[e^{ixk} \partial \bar{\partial}^{-1}(q e^{-ixk} \partial^{-1}(e^{ixk} \partial \bar{\partial}^{-1} q))] . \end{aligned} \tag{3.7}$$

Since ω_0 and ω_1 are defined for all k except $k = 0$, we may drop the function Φ_K in (3.3) with $j = 0$ and define

$$\tilde{T}_q^{(0)} = \int_\Omega e^{ixk} q(x)\omega_0(x, k)dx .$$

One sees immediately that $\mathscr{F}^{-1}\tilde{T}_q^{(0)} - \mathscr{F}^{-1}T_q^{(0)}$ is a C^∞ function. Therefore, we need only to show

$$\mathscr{F}^{-1}T_q^{(1)}, \mathscr{F}^{-1}\tilde{T}_q^{(0)} \in C^\alpha(\mathbb{R}^2), \quad \forall\alpha, \quad 0 \leqq \alpha < 1 . \tag{3.8}$$

In the rest of this section we prove (3.8). In what follows, we denote $z = x_1 + ix_2, z \in \mathbb{C}, \tilde{z} = (x_1, x_2) \in \mathbb{R}^2$. We divide the rest of this section into two parts. In the first part we prove (3.8) with $j = 0$ and in the second part we treat the case $j = 1$.

Part I. Proof of (3.8) *with* $j = 0$. We first prove a lemma.

Lemma 3.3. *There exists a constant C such that*

$$\mathscr{F}^{-1}\tilde{T}_q^{(0)}(z) = \frac{C}{\bar{z}} * [q\bar{\partial}^{-1}q - (\partial\bar{\partial}^{-1}q)\partial^{-1}q] , \tag{3.9}$$

where $*$ *denotes convolution in* $\mathbb{R}^2$.

Proof. From (3.6) we deduce that

$$\mathscr{F}^{-1}\tilde{T}_q^{(0)} = \mathscr{F}^{-1}\left[\frac{1}{k_2 + ik_1}\int_\Omega e^{ik\tilde{x}} q\bar{\partial}^{-1}qdx\right]$$

$$+ \mathscr{F}^{-1}\left[\frac{1}{k_2 + ik_1}\int_\Omega q\partial^{-1}(e^{ik\tilde{z}}\partial\bar{\partial}^{-1}q)dz\right] . \tag{3.10}$$

Clearly,

$$\mathscr{F}^{-1}\left[\frac{1}{k_2 + ik_1}\int_\Omega e^{ik\tilde{x}} q\bar{\partial}^{-1}qdz\right] = \frac{C}{\bar{z}} * (q\bar{\partial}^{-1}q) . \tag{3.11}$$

By (2.10), we have

$$\int_\Omega q\partial^{-1}(e^{ik\tilde{z}}\partial\bar{\partial}^{-1}q)dz = -\frac{1}{\pi}\int_\Omega q(z)\int_{R^2}\frac{e^{ik\tilde{\xi}}(\partial\bar{\partial}^{-1}q)(\xi)}{\bar{\xi} - \bar{z}}d\xi dz .$$

Since $\partial\bar{\partial}^{-1}q \in L^p(\mathbb{R}^2)$ for any $p > 1$ (one can see this by using (2.14) and (2.15)), it follows that,

$$\int_\Omega |q(z)| \int_{\mathbb{R}^2}\left|\frac{(\partial\bar{\partial}^{-1}q)(\xi)}{\bar{\xi} - \bar{z}}\right| d\xi dz \tag{3.12}$$

is finite. Thus, by Fubini's theorem,

$$\begin{aligned}\int_\Omega q\partial^{-1}(e^{ik\tilde{z}}\partial\bar{\partial}^{-1}q)dz &= -\frac{1}{\pi}\int_{\mathbb{R}^2} e^{ik\tilde{\xi}}(\partial\bar{\partial}^{-1}q(\xi))\int_\Omega\frac{q(z)}{\bar{\xi} - \bar{z}}dzd\xi \\ &= -\int_{\mathbb{R}^2} e^{ik\tilde{\xi}}(\partial\bar{\partial}^{-1}q(\xi))(\partial^{-1}q)(\xi)d\xi \\ &= -\mathscr{F}((\partial\bar{\partial}^{-1}q)\partial^{-1}q) .\end{aligned} \tag{3.13}$$

Therefore,

$$\mathscr{F}^{-1}\left[\frac{1}{k_2+ik_1}\int_\Omega q\partial^{-1}(e^{ik\bar{z}}\partial\bar{\partial}^{-1}q)dz\right] = -\mathscr{F}^{-1}\left[\frac{1}{k_2+ik_1}\mathscr{F}(\partial\bar{\partial}^{-1}q)\partial^{-1}q\right]$$

$$= -C\frac{1}{\bar{z}}*((\partial\bar{\partial}^{-1}q)\partial^{-1}q)\,. \tag{3.14}$$

Combining (3.11) and (3.14) with (3.10) we get (3.9). □

Using (2.11) and (2.14) we have that $\partial^{-1}q$ is bounded and thus

$$q\bar{\partial}^{-1}q - (\partial\bar{\partial}^{-1}q)\partial^{-1}q \in L^p(\mathbb{R}^2), \quad \forall p > 1\,.$$

Combining this with the next lemma we get the desired result.

Lemma 3.4. *Let $p_0 \in \mathbb{R}$, $2 < p_0 < \infty$. Let $f \in L^p(\mathbb{R}^2)$ $\forall p$, $1 < p < p_0$. Then $\frac{1}{z}*f(z) \in C^\alpha(\mathbb{R}^2)$ for $\alpha = 1 - \frac{2}{p_0}$.*

Proof. Using Hölder's inequality and the hypothesis one shows easily that the function

$$F(z) = \frac{1}{z}*f(z) = \int_{\mathbb{R}^2}\frac{f(\xi)}{z-\xi}d\xi$$

is well defined for each z. Consider

$$F(z+h) - F(z) = -\int_{\mathbb{R}^2}\frac{hf(\xi)}{(z+h-\xi)(z-\xi)}d\xi\,.$$

Let δ be a positive number in (0, 1). Using Hölder's inequality we get

$$|F(z+h)-F(z)| \leqq |h|\left(\int_{\mathbb{R}^2}\frac{d\xi}{|z-\xi+h|^{1+\delta}|z-\xi|^{1+\delta}}\right)^{\frac{1}{1+\delta}}\|f\|_{L^{\frac{1+\delta}{\delta}}(\mathbb{R}^2)}\,.$$

Now

$$\int_{\mathbb{R}^2}\frac{d\xi}{|z-\xi+h|^{1+\delta}|z-\xi|^{1+\delta}} = \int_{\mathbb{R}^2}\frac{d\xi}{|\xi-h|^{1+\delta}|\xi|^{1+\delta}}$$

$$= \frac{1}{|\xi|^{1+\delta}}*\frac{1}{|\xi|^{1+\delta}} = \mathscr{F}^{-1}\left(\left(\mathscr{F}\left(\frac{1}{|\xi|^{1+\delta}}\right)\right)^2\right)$$

$$= C\mathscr{F}^{-1}\left(\left(\frac{1}{|\xi|^{1-\delta}}\right)^2\right) = C\mathscr{F}^{-1}\left(\frac{1}{|\xi|^{2-2\delta}}\right) = C\frac{1}{|h|^{2\delta}}\,.$$

Then it follows that

$$|F(z+h)-F(z)| \leqq C\|f\|_{L^{\frac{1+\delta}{\delta}}(\mathbb{R}^2)}|h|^{1-\frac{2\delta}{1+\delta}}\,.$$

This shows that $F \in C^{1-\frac{2\delta}{1+\delta}}(\mathbb{R}^2)$, $0 < \delta < 1$. Now, let $\dfrac{1+\delta}{\delta} = p_0 > 2$, then $\delta = \dfrac{1}{p_0 - 1}$, hence $1 - \dfrac{2\delta}{1+\delta} = 1 - \dfrac{2}{p_0}$ concluding the proof.

Part II. Proof of (3.8) *with* $j = 1$. Using (3.7), we get

$$\mathscr{F}^{-1} T^{(1)} = L_1 - L_2 - L_3 - L_4 ,$$

where L_j is the function of k given below, $j = 1, 2, 3, 4$.

$$L_1 = \mathscr{F}^{-1}\left[(k_2 + ik_1)^{-2} \Phi_k \int_\Omega e^{ik\bar{z}} q\bar{\partial}^{-1}(q\bar{\partial}^{-1} q)dx \right], \tag{3.16}$$

$$L_2 = \mathscr{F}^{-1}\left[(k_2 + ik_1)^{-2} \Phi_K \int_\Omega e^{ik\bar{z}} q\bar{\partial}^{-1}(qe^{-ik\bar{z}} q\partial^{-1}(e^{ik\bar{z}}\partial\bar{\partial}^{-1} q))dx \right], \tag{3.17}$$

$$L_3 = \mathscr{F}^{-1}\left[(k_2 + ik_1)^{-2} \Phi_K \int_\Omega q\partial^{-1}(e^{ik\bar{z}}\partial\bar{\partial}^{-1}(q\bar{\partial}^{-1} q))dx \right], \tag{3.18}$$

$$L_4 = \mathscr{F}^{-1}\left[(k_2 + ik_1)^{-2} \Phi_K \int_\Omega q\partial^{-1}(e^{ik\bar{z}}\partial\bar{\partial}^{-1}(qe^{-ik\bar{z}}\partial^{-1}(e^{ik\bar{z}}\partial\bar{\partial}^{-1} q)))dx \right], \tag{3.19}$$

We shall first show that

$$L_j \in H^2(\mathbb{R}^2), \quad j = 1, 2, 3 . \tag{3.20}$$

This implies by Sobolev's embedding theorem our result. Since $\Phi_K(k) = 0$ for $|k| < K$, it follows that (3.20) is a consequence of the following three results:

$$I_1(k) = \int_\Omega e^{ik\bar{z}} q\bar{\partial}^{-1}(q\bar{\partial}^{-1} q)dx \in L^2(\mathbb{R}^2) , \tag{3.21}$$

$$I_2(k) = \int_\Omega e^{ik\bar{z}} q\bar{\partial}^{-1}(qe^{-ik\bar{z}}\partial^{-1}(e^{ik\bar{z}}\partial\bar{\partial}^{-1} q))dx \in L^2(\mathbb{R}^2) , \tag{3.22}$$

$$I_3(k) = \int_\Omega q\partial^{-1}(e^{ik\bar{z}}\partial\bar{\partial}^{-1}(q\bar{\partial}^{-1} q))dx \in L^2(\mathbb{R}^2) . \tag{3.23}$$

Since $q\bar{\partial}^{-1} q$ and $\bar{\partial}^{-1}(q\bar{\partial}^{-1} q)$ are bounded function according to (2.12), one can use the same method as the one given in Part I to show that

$$\mathscr{F}^{-1}(I_1 - I_3) = q\bar{\partial}^{-1}(q\bar{\partial}^{-1} q) + [\partial\bar{\partial}^{-1}(q\bar{\partial}^{-1} q)]\partial^{-1}(q\bar{\partial}^{-1} q) \in L^2(\mathbb{R}^2) .$$

Thus we need only to prove (3.22).

We have

$$I_2 = \frac{1}{\pi^2} \int_\Omega e^{ik\bar{\tau}} q(\tau) \int_\Omega \frac{e^{-ik\bar{\tau}} q(\tau)}{\tau - z} \int_{\mathbb{R}^2} e^{ik\bar{\eta}} \frac{(\partial\bar{\partial}^{-1} q)(\eta)}{\bar{\eta} - \bar{\tau}} d\eta \, d\tau \, dz .$$

By making the change of variables in η: $\eta \to \eta - z + \tau$, we get

$$I_2 = \frac{1}{\pi^2} \int_\Omega q(z) \int_\Omega \frac{q(\tau)}{\tau - z} \int_{\mathbb{R}^2} e^{ik\bar{\eta}} \frac{(\partial\bar{\partial}^{-1} q)(\eta + \tau - z)}{\bar{\eta} - \bar{z}} d\eta \, d\tau \, dz .$$

Using Propositions 2.3 and 2.4 one checks easily that we can use Fubini's theorem to exchange the order of integrations in I_2. Therefore,

$$I_2 = \frac{1}{\pi^2} \int_{\mathbb{R}^2} e^{ik\bar{\eta}} \left[\int_\Omega \frac{q(z)}{\bar{\eta} - \bar{z}} \int_\Omega \frac{q(\tau)}{\tau - z} (\partial\bar{\partial}^{-1} q)(\eta + \tau - z)\, d\tau\, dz \right] d\eta\,. \tag{3.24}$$

Using (2.14) and Hölder's inequality we have

$$\left| \int_\Omega \frac{q(\tau)}{\tau - z} (\partial\bar{\partial}^{-1} q)(\eta + \tau - z)\, d\tau \right| \leqq \left(\int_\Omega \left| \frac{q(\tau)}{\tau - z} \right|^p dz \right)^{\frac{1}{p}} \left(\int_\Omega |\partial\bar{\partial}^{-1} q|^q (\eta + \tau - z)\, d\tau \right)^{\frac{1}{q}}$$

$$\leqq \left(\int_\Omega \left| \frac{q(\tau)}{\tau - z} \right|^p d\tau \right)^{\frac{1}{p}} \| \partial\bar{\partial}^{-1} q \|_{L^q(\mathbb{R}^2)}\,, \tag{3.25}$$

where $1 < p < 2$, and $\frac{1}{p} + \frac{1}{q} = 1$. It is easy to see that the right side of (3.25) is uniformly bounded in z and η which implies that $\mathscr{F}^{-1} I_2 \in L^2_{\text{loc}}(\mathbb{R}^2)$. To show that $\mathscr{F}^{-1} I_2 \in L^2(\mathbb{R}^2)$ and thus $I_2 \in L^2(\mathbb{R}^2)$, we use (2.15). Since $\tau, z \in \Omega$, it follows from (2.15) that for $|\eta|$ large,

$$\left| \int_\Omega \frac{q(\tau)}{\tau - z} (\partial\bar{\partial}^{-1} q)(\eta + \tau - z)\, d\tau \right| \leqq \frac{C}{|\eta|^2}$$

for some constant $C > 0$. Therefore,

$$|\mathscr{F}^{-1} I_2(\eta)| \leqq C \int_\Omega \left| \frac{q(z)}{\bar{\eta} - \bar{z}} \right| dz \frac{1}{|\eta|^2} \leqq \frac{C}{|\eta|^3}$$

for large $|\eta|$, concluding that $\mathscr{F}^{-1} I_2 \in L^2(\mathbb{R}^2)$.

We now prove that $I_4 \in C^\alpha(\mathbb{R}^2)$, $0 \leqq \alpha < 1$. Write

$$I_4 = \mathscr{F}^{-1}((k_2 + ik_1)^{-2} \Phi_K P)\,, \tag{3.26}$$

where

$$P(k) = \int_\Omega q \partial^{-1}(e^{ik\bar{z}} \partial\bar{\partial}^{-1}(q e^{-ik\bar{z}} \partial^{-1}(e^{ik\bar{z}} \partial\bar{\partial}^{-1} q)))\, dz$$

$$= \frac{1}{\pi} \int_\Omega q(z) \int_{\mathbb{R}^2} \frac{e^{ik\bar{\tau}}}{\bar{\tau} - \bar{z}} \partial\bar{\partial}^{-1}(q e^{-ik\bar{\tau}} \partial^{-1}(e^{ik\bar{\tau}} \partial\bar{\partial}^{-1} q))(\tau)\, d\tau\, dz\,.$$

Since $\bar{\partial}^{-1}(q e^{-ik\bar{\tau}} \partial^{-1}(e^{ik\bar{\tau}} \partial\bar{\partial}^{-1} q)) \in L^p$ for $p > 1$, we can use Fubini's theorem to exchange dz and $d\tau$. Hence,

$$P(k) = -\frac{1}{\pi} \int_{\mathbb{R}^2} e^{ik\bar{\tau}} \partial\bar{\partial}^{-1}(q e^{-ik\bar{\tau}} \partial^{-1}(e^{ik\bar{\tau}} \partial\bar{\partial}^{-1} q))(\tau) \int_\Omega \frac{q(z)}{\bar{\tau} - \bar{z}}\, dz\, d\tau$$

$$= -\int_{\mathbb{R}^2} e^{ik\bar{\tau}} (\partial^{-1} q)(\tau) \partial\bar{\partial}^{-1}(q e^{-ik\bar{\tau}} \partial^{-1}(e^{ik\bar{\tau}} \partial\bar{\partial}^{-1} q))(\tau)\, d\tau\,. \tag{3.27}$$

Since $\partial\bar{\partial}^{-1}$ is a singular integral operator, we cannot use Fubini's theorem. We will change variables and integrate by parts.

Using integration by parts, we can write

$$P(k) = \int_{\mathbb{R}^2} e^{ik\bar{\tau}} q(\tau)\bar{\partial}^{-1}(qe^{-ik\bar{\tau}}\partial^{-1}(e^{ik\bar{\tau}}\partial\bar{\partial}^{-1}q))d\tau$$

$$+ (k_2 + ik_1) \int_{\mathbb{R}^2} e^{ik\bar{\tau}}(\partial^{-1}q)\bar{\partial}^{-1}(qe^{-ik\bar{\tau}}\partial^{-1}(e^{ik\bar{\tau}}\partial\bar{\partial}^{-1}q))d\tau$$

$$= \Pi_1 + (k_2 + ik_1)\Pi_2 . \tag{3.28}$$

We can perform integration by parts here because the second integral in (3.27) and the two integrals in (3.28) converge in $L^1(\mathbb{R}^2)$. Also, $(\partial^{-1}q)(\tau)$ and $(\bar{\partial}^{-1}(qe^{-ik\bar{\tau}}\partial^{-1}(e^{ik\bar{\tau}}\partial\bar{\partial}^{-1}q)))(\tau)$ behave like $\frac{1}{|\tau|^2}$ as $|\tau|$ tends to ∞. (See Propositions 2.3 and 2.4.)

Substituting (3.28) into (3.26) yields

$$I_4 = \mathscr{F}^{-1}((k_2 + ik_1)^{-2}\Phi_K\Pi_1) + \mathscr{F}^{-1}((k_2 + ik_1)^{-1}\Phi_K\Pi_2)$$

$$= N_1 + N_2 . \tag{3.29}$$

We shall prove $N_1 \in C_0^\alpha(\mathbb{R}^2)$, $0 \leqq \alpha < 1$ by showing that

$$\Pi_1 \in L^2(\mathbb{R}^2) . \tag{3.30}$$

Using Fubini's theorem, and the change of variables in s: $s \to s - \tau + \eta$, we get

$$\Pi_1 = \frac{1}{\pi^2}\int_\Omega e^{ik\bar{\tau}}q(\tau)\int_\Omega e^{-ik\bar{\eta}}\frac{q(\eta)}{\eta - \tau}\int_{\mathbb{R}^2} e^{ik\bar{s}}\left(\frac{(\partial\bar{\partial}^{-1}q)(s)}{\bar{s} - \bar{\eta}}\right)dx\,d\eta\,d\tau$$

$$= \frac{1}{\pi^2}\int_\Omega q(\tau)\int_\Omega \frac{q(\eta)}{\eta - \tau}\int_{\mathbb{R}^2}\frac{e^{ik\bar{s}}\partial\bar{\partial}^{-1}q(s - \tau + \eta)}{\bar{s} - \bar{\tau}}ds\,d\eta\,d\tau$$

$$= \frac{1}{\pi^2}\int_{\mathbb{R}^2} e^{ik\bar{s}}\int_\Omega\frac{q(\tau)}{\bar{s} - \bar{\tau}}\int_\Omega\frac{q(\eta)(\partial\bar{\partial}^{-1}q)(s - \tau + \eta)}{\eta - \tau}d\eta\,d\tau\,ds . \tag{3.31}$$

By making the change of variables $\eta + s - \tau \to \eta$, we get

$$\Pi_1 = \mathscr{F}\left[\frac{1}{\pi^2}\int_\Omega\frac{q(\tau)}{\bar{s} - \bar{\tau}}\int_\Omega\frac{q(\eta - s + \tau)(\partial\bar{\partial}^{-1}q)(\eta)}{\eta - s}d\eta\,d\tau\right].$$

Now applying Fubini's theorem we obtain

$$\Pi_1 = \mathscr{F}\left[\frac{1}{\pi^2}\int_{\mathbb{R}^2}\frac{(\partial\bar{\partial}^{-1}q)(\eta)}{\eta - s}\int_\Omega\frac{q(\tau)q(\eta - s + \tau)}{\bar{s} - \bar{\tau}}d\tau\,d\eta\right].$$

Therefore, it suffices to show

$$\int_{\mathbb{R}^2}\frac{(\partial\bar{\partial}^{-1}q)(\eta)}{\eta - s}\int_\Omega\frac{q(\tau)q(\eta - s + \tau)}{\bar{s} - \bar{\tau}}d\tau\,d\eta \in L^2(\mathbb{R}^2) . \tag{3.32}$$

Since, by (2.12),

$$\left|\int_\Omega\frac{q(\tau)q(\eta - s + \tau)}{\bar{s} - \bar{\tau}}d\tau\right| \leqq \frac{C}{1 + |s|}\|q\|^2_{L^\infty_\Omega} ,$$

and thus, by Propositions (2.3) and (2.4) we conclude

$$|\mathscr{F}^{-1}(\Pi_1)(s)| \leqq \frac{C}{(1+|s|)(1+|s|)^{1+\delta}} \|q\|^2_{L^\infty_\Omega} \|\partial\bar{\partial}^{-1} q\|_{L^p_{\delta+1}(\mathbb{R}^2)}$$

$$\leqq \frac{C}{(1+|s|)^{2+\delta}} \|q\|^3_{L^\infty_\Omega},$$

where $p > 2$, $-1 < \delta - 1 + \frac{2}{p} < 0$. This estimate gives (3.32). Finally, since $N_2 - \mathscr{F}^{-1}((k_2 + ik_1)^{-1}\Pi_2) \in C^\infty$, we need only to consider

$$\mathscr{F}^{-1}((k_2 + ik_1)^{-1}\Pi_2) = C\left(\frac{1}{\bar{s}} * (\mathscr{F}^{-1}\Pi_2)(s)\right).$$

By virtue of Lemma 3.4, we need only to show $\mathscr{F}^{-1}\Pi_2 \in L^p(\mathbb{R}^2)$ for any $p > 1$. As in (3.31), we get

$$(\mathscr{F}^{-1}\Pi_2)(s) = \frac{2}{\pi^2} \int_{\mathbb{R}^2} \frac{(\partial\bar{\partial}^{-1}q)(\eta)}{\eta - s} \int_\Omega \frac{(\partial^{-1}q)(\tau)q(\eta - s + \tau)}{\bar{s} - \bar{\tau}} d\tau d\eta .$$

By making the change of variables $\tau + \eta - s \to \tau$, we obtain

$$\mathscr{F}^{-1}(\Pi_2)(s) = \frac{1}{\pi^2} \int_{\mathbb{R}^2} \frac{(\partial\bar{\partial}^{-1}q)(\eta)}{\eta - s} \int_\Omega \frac{q(\tau)(\partial^{-1}q)(\tau + s - \eta)}{\bar{\eta} - \bar{\tau}} d\tau d\eta .$$

By (2.12)

$$\left| \int_\Omega \frac{q(\tau)(\partial^{-1}q)(\tau + s - \eta)}{\bar{\eta} - \bar{\tau}} d\tau \right| \leqq \frac{C\|q\|^2_{L^\infty_\Omega}}{1 + |\eta|} \sup_{\tau\in\Omega} |(\partial^{-1}q)(\tau + s - \eta)|$$

$$\leqq \frac{C\|q\|^2_{L^\infty_\Omega}}{(1+|\eta|)(1+|s-\eta|)} \leqq \frac{C\|q\|^2_{L^\infty_\Omega}}{1+|s|}.$$

Thus, by (2.11)

$$|\mathscr{F}^{-1}(\Pi^2)(s)| \leqq \frac{C}{(1+|s|)(1+|s|)^{1+\delta}} \|q\|_{L^\infty_\Omega} \|\partial\bar{\partial}^{-1}q\|_{L^{p'}_{\delta+1}(\mathbb{R}^2)},$$

where $-1 < \delta - 1 + \frac{2}{p'} < 0$, $p' > 2$. Then we can set $\delta = 0$. Thus $\mathscr{F}^{-1}(\Pi_2) \in L^p(\mathbb{R}^2)$, $\forall p > 1$, concluding the result. □

4. Proofs of Theorems B and C and Remarks

This reduction of Theorems B and C to Theorem A is well known in the case $n > 2$ ([N, St]). The same method of proof applies in the 2 dimensional case. We indicate the main steps here for the purpose of completeness.

Assume $\operatorname{supp} q_i \subseteq B(0, R)$ with $R > 0$. Here $B(0, R) = \{x \in \mathbb{R}^2, |x| < R\}$. We may choose R so that zero is not a Dirichlet eigenvalue for $\Delta - q_j, j = 1, 2$.

The outgoing Green's kernel for $-\Delta + q$ has the asymptotic expansion for large $|x|$:

$$G_q(x, x_0, \lambda) = \frac{e^{i\lambda|x|}}{|x|^{\frac{1}{2}}} \psi_q(\lambda, x_0, -\theta) + 0(|x|^{-\frac{3}{2}})$$

with $\theta = \dfrac{x}{|x|}$ and ψ_q the outgoing eigenfunction. This can be seen by using the large $|x|$ asymptotics of the free outgoing Green's kernel (associated to the potential zero) which is given by the Hankel function (see for example [A-S]):

$$G_0(x, x_0, \lambda) = \frac{i}{4} H_0^{(1)}(|x - x_0|\lambda)$$

and writing an integral equation for the outgoing Green's kernel in terms of the free one (see [St] in the 3 dimensional case).

Now if $a_{q_1}(\lambda, \theta, \omega) = a_{q_2}(\lambda, \theta, \omega)$ for a fixed λ then,

$$G_{q_1}(x, x_0, \lambda) - G_{q_2}(x, x_0, \lambda) = 0(|x|^{-\frac{3}{2}}|x_0|^{-\frac{3}{2}})$$

and

$$(\Delta_x - \lambda^2)(G_{q_1}(x, x_0, \lambda) - G_{q_2}(x, x_0, \lambda)) = 0, \quad |x|, |x_0| \geqq R .$$

Therefore by Rellich's Lemma we conclude

$$G_{q_1}(x, x_0, \lambda) = G_{q_2}(x, x_0, \lambda), \quad |x|, |x_0| \geqq R . \tag{4.1}$$

Define the single layer potential

$$S_{q_i, \lambda} f(x) = \int_{\partial B(0,R)} G_{q_i}(x, x_0, \lambda) f(x_0) dS_{x_0} .$$

Then (4.1) implies that

$$S_{q_1, \lambda} = S_{q_2, \lambda} .$$

Therefore, by formula (1.40) in [N] (also valid in two dimensions) we get

$$\Lambda_{q_1 - \lambda^2} = \Lambda_{q_2 - \lambda^2} . \tag{4.2}$$

Now using Theorem A we conclude Theorem B.

The reduction of Theorem C to Theorem A follows along the same lines (see [N] in the case $n > 2$). In fact, if we denote

$$q_j = -\lambda^2 \left(\frac{1}{c_j^2} - \frac{1}{c_0^2} \right), \quad j = 1, 2 ,$$

then

$$\mathcal{G}_{c_j}(x, x_0, \lambda) = G_{q_j}\left(x, x_0, \frac{\lambda^2}{c_0^2} \right) .$$

Thus, by the above arguments, the hypothesis of Theorem C implies

$$\Lambda_{-\frac{\lambda^2}{c_1^2}} = \Lambda_{-\frac{\lambda^2}{c_2^2}} .$$

(By replacing Ω by a large ball we may assume that 0 is not an eigenvalue for $\Delta + \frac{\lambda^2}{c_j^2}, j = 1, 2$ on Ω). Theorem C then follows from (4.3) and Theorem A.

We remark that the proof given in Sects. 2 and 3 carries over to the case of L^p potentials. The analog of the exponential growing solutions (2.1) for a L^p potential in dimension two has been constructed by Ikehata [I]. Using these solutions and Lemma 3.4 one can show the following analog of Theorem A which we state without proof.

Theorem F. *Let* $q_i \in L^p(\Omega), 2 < p < \infty$, *with* 0 *not a Dirichlet eigenvalue of* $-\Delta + q_i, i = 1, 2$. *Assume*

$$\Lambda_{q_1} = \Lambda_{q_2} .$$

Then

$$q_1 - q_2 \in C^{1-\frac{2}{p}}(\bar{\Omega}) .$$

Analog results to Theorems B and C can also be stated in a similar fashion.

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